Cabra's Scientific Banksy

The Story of William Rowan Hamilton and Quaternions

By **James Hayes**

Of the numerous bridges over the Royal Canal, there are but two with their own Wikipedia page. The first is Mullen Bridge in Maynooth, County Kildare. Visitors to its page are greeted with a warning "This article needs additional citations for verification". At the foot of the article, which consists of four lines, there is a notice which informs visitors that the page is a stub - "an article too short to provide encyclopedic coverage of a subject", a mere acknowledgement of its existence. The other bridge is Broom Bridge in Cabra.

As bridges go, this one is not particularly large or imposing, nor is it an engineering marvel. In fact, were it not for the small commemorative plaque on the bridge's west side, one would be forgiven for assuming there was little of note about this bridge. And how wrong that assumption would be! This squat stone bridge on Dublin's Northside is of huge importance not just in Ireland's scientific heritage, but the world's. Some regard it as the birthplace of modern algebra, a place where mathematics was liberated from dogmatic conventions that had existed for centuries. And it all stemmed from a piece of graffiti!

It is a story that begins five Luas stops away from Broombridge, on Dominick Street in the city's centre. At number 29 (which has since been renumbered number 36), William Rowan Hamilton was born at midnight on the 3rd August 1805. His brilliance was apparent from a young age and at the age of three he had been shipped off to live with his uncle, James Hamilton, in Trim.

James was a distinguished linguist and it was he who dictated the young William's education. Shortly before William's tenth birthday his uncle wrote that "His thirst for the Oriental languages is unabated. He is now master of most, indeed of all except the minor and comparatively provincial ones, The Hebrew, Persian and Arabic are about to be confirmed by the superior and intimate acquaintance with the Sanskrit, in which he is already a proficient. The Chaldee and Syriac he is grounded in, also the Hindustanee, Malay, Mahratta, Bengali and others. He is about to commence the Chinese..."

By the age of thirteen he could lay claim to the title of hyperpolyglot, the term used by linguists to describe a person who can speak at least a dozen languages. A year later, upon hearing of the visit of the Persian Ambassador to Dublin, he composed a welcome poem in Persian for him. To his disappointment, the ambassador "much regretted that on account of a bad headache he was unable to receive me (Hamilton) personally". However, it was not through these linguistic feats that Hamilton was set on the path to scientific greatness.

These days, there is little of note about Cabot, Vermont, a sleepy New England town situated just over an hour's drive from the border with Quebec. This hamlet of just over 1,000 people houses a large creamery of the same name, whose cheeses have won several awards (Cabot

Monterrey Jack was honoured by the American Cheese Society in 2008). It is also the birthplace of the current coach of the Angolan national basketball team, a man named William Voigt. But in 1813, Cabot was the home of a calculating prodigy by the name of Zerah Colburn, whose astonishing aptitude at feats of mental arithmetic made him something of a minor celebrity.

Zerah's father, Abia, capitalized on the boy's ability by touring him as a novelty, first in the United States, and then in Europe. It was while in Dublin, in September of 1813, that Hamilton and Colburn crossed paths. The two prodigies were pitted against each other in a battle of the wits. Hamilton was duly defeated by Colburn, and decided from that day forth to devote more time to mathematics and focus on languages less, a decision that put him firmly on the trajectory to becoming Ireland's foremost mathematician. Colburn went on to become professor of languages at Dartmouth College (a member of the prestigious Ivy League of US universities), and died in 1839 aged just 34. Curiously, he wrote in his autobiography that by adulthood he had completely lost his powers of mental arithmetic.

Hamilton's illustrious academic career began with a clear indication of what was to come. On the 7th July 1823, a month shy of his eighteenth birthday, Hamilton sat the entrance exams for Trinity College in Dublin. Despite his lack of any formal schooling (his education was received solely from his uncle and his own private study), he comfortably passed the exams, with the top results out of the one hundred or so candidates. His exploits as an undergraduate are legendary, winning countless prizes and obtaining the highest honours in both classics and mathematics. But even these considerable achievements pale in comparison to the scarcely believable fashion in which his undergraduate career at Trinity concluded.

In 1827, the then Professor of Astronomy at Trinity (a job that came with the rather grandiose title Royal Astronomer of Ireland), John Brinkley resigned his professorship to become the Anglican Bishop of Cloyne, in County Cork. The vacancy was duly advertised. Among the applicants was George Biddell Airy, a man who later went on to become Astronomer Royal of England. The Governing Board of the University, following a short period of reflection, decided to pass over all the applicants and unanimously elected Hamilton to the professorship. Hamilton, still an undergraduate and at this point 22 years old, had not even applied for the job! Hamilton accepted the post, and took up residence at the Observatory in Dunsink, where he lived for nearly 40 years, up to his death in 1865.

It was from Dunsink Observatory that Hamilton set out with his wife on the fateful day of 16th of October 1843, bound for the Royal Irish Academy (then situated on Grafton Street). What happened next is the stuff of mathematical legend. For the 15 years or so prior, Hamilton had been grappling with a conundrum. He sought to extend the complex numbers, which can be represented as points on a plane (two dimensions), to points in space (three dimensions).

The problem for Hamilton was that while addition and subtraction of these "triples" was rather straightforward, he could not define multiplication and division operations for said triples. But on this autumn day, walking along the Royal Canal, it dawned on him. In the delirium of this flash of genius, he carved, using a penknife, the formula for his invention into the stone of Broom Bridge:

$$i^2 = j^2 = k^2 = ijk = -1$$

Hamilton had realised that to achieve this goal of multiplying and dividing three-dimensional points, he had to go one better, and use four dimensions. He christened his discovery "quaternions" as a result. Hamilton's carvings represent the basic rules of multiplication for these quadruples. It was a discovery that sent reverberations throughout the mathematical world and whose implications and applications survive to this day.

"Why is a raven like a writing desk?" The Mad Hatter poses this riddle to Alice at his tea party in Lewis Carroll's Alice in Wonderland. When Alice later asks Hatter what the answer was, he replies "I haven't the slightest idea". This scene is just one of the many in the novel where Carroll, himself a mathematician, offers a narrative on how nonsensical he felt 'modern' mathematics was becoming. Carroll felt the increasing abstraction that was taking place at the time was destroying the subject, and that mathematicians should return to the concrete roots of the subject in Euclidean geometry. Of particular concern to Carroll was the discovery of non-commutative algebras - of which Hamilton's quaternions were the first!

Hamilton's quaternions ditched the commutativity that all prior number systems used. This meant that in Hamilton's system of quaternions the order of multiplying two numbers mattered – x times y was no longer equal to y times x. This was ludicrous to Carroll, who felt algebra was fundamentally based in numbers, where for example, 5 times 7 was always equal to 7 times 5. Hamilton had the last laugh however, as this abstraction proved itself to be more than a fad and caught on, paving the way for the creation of all manner of useful and interesting algebraic systems. In fact, one might say that Hamilton's work liberated mathematics from the traditional views of Carroll and his ilk that had been held for centuries and set mathematics on a new course.

But Hamilton's discovery was good for a lot more than satirizing in 19th century fiction. While the quaternion's early popularity was quickly supplanted by the significantly easier to use vector, it experienced something of a revival in the second half of the 20th century and the start of the 21st. Quaternions are now used to represent rotations in a whole host of real-world applications. Chief among them is their use in NASA's flight navigation software. NASA originally used rotational matrices to achieve rotations, but they came with a serious flaw: gimbal lock.

When two gimbals become aligned (such as when a spacecraft pitches up 90 degrees), a degree of freedom is lost. The three gimbals can still rotate freely, but as two of them are parallel, it means that there will be one axis which rotation about cannot be accommodated. The early Apollo missions incorporated a gimbaled system in their navigational system, the primary guidance, navigation and control system (PGNCS, which is apparently pronounced 'pings'). The inertial measurement unit (IMU) component of the system used gimbals as part of its system to keep track of the spacecraft's velocity and position. However this meant that gimbal lock could result in the spacecraft losing track of its position in space.

A well-documented incident of this gimbal lock took place on the Apollo 11 lunar mission, where the spacecraft's navigation system froze as a result, almost resulting in catastrophe. Quaternions, however, do not have this same problem with gimbal lock. NASA promptly adopted Hamilton's brainchild, using quaternions to represent rotations for the 1981 launch of the Space Shuttle, and they have never looked back. An added benefit is that using quaternions for rotations is much less demanding on computers. The usefulness of the quaternion isn't just confined to space flight, however - they also have applications in computer graphics and computer vision (helping computers process information from images, of particular interest in self-driving cars).

While all traces of Hamilton's act of minor vandalism are gone from Broom Bridge, the legacy of this piece of mathematical graffiti lives on. The 16th of October each year is celebrated as Hamilton Day. Chief among the festivities is the Hamilton walk. Organised by the Mathematics Department in NUI Maynooth, the walk retraces Hamilton's steps on that fateful day in 1843, starting at Dunsink Observatory and finishing at the bridge. It is a walk like no other, one that attracts students and Fields Medallists alike, along with the occasional Nobel Prize winner (usually in Physics). They all come to commemorate the same thing - an Irishman's moment of Banksy-esque brilliance and the enormous scientific legacy that came with it.